

# $\mathcal{O}(\alpha_S^5 m)$ quarkonium $1S$ spectrum in large- $\beta_0$ approximation and renormalon cancellation

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## Abstract

Presently the quarkonium spectrum, written in terms of the quark  $\overline{\text{MS}}$ -mass, is known at  $\mathcal{O}(\alpha_S^3 m)$  accuracy. We point out that in order to achieve  $\mathcal{O}(\alpha_S^4 m)$  accuracy it is sufficient to calculate further (I) the  $\mathcal{O}(\alpha_S^4 m)$  relation between the  $\overline{\text{MS}}$ -mass and the pole-mass, and (II) the binding energy at  $\mathcal{O}(\alpha_S^5 m)$  in the large- $\beta_0$  approximation. We calculate the latter correction analytically for the  $1S$ -state and study its phenomenological implications.

# 1 Introduction

In recent years theory necessary for precise description of a heavy quarkonium such as bottomonium or (remnant of) toponium has developed significantly. In particular, development in the computational technology of higher order corrections to the quarkonium energy spectrum [1, 2] and subsequent discovery of the renormalon cancellation [3, 4] enabled accurate determinations of the  $\overline{\text{MS}}$ -mass of the bottom quark [1, 5, 2, 6] and (in the future) of the top quark [7]. In these determinations a major part is played by the spectrum (mass) of the  $1S$ -state quarkonium.\*

It is legitimate to consider that the present perturbative calculation of the quarkonium spectrum, when expressed in terms of the quark  $\overline{\text{MS}}$ -mass, has a *genuine accuracy* at  $\mathcal{O}(\alpha_S^3 m)$ . In fact, in formal power countings, the last known term in the relation between the  $\overline{\text{MS}}$ -mass and the pole-mass of a quark is  $\mathcal{O}(\alpha_S^3 m)$ , while the last known term of the binding energy (measured from twice of the quark pole-mass) is  $\mathcal{O}(\alpha_S^4 m)$ . The former term includes in addition to a genuine  $\mathcal{O}(\alpha_S^3 m)$  part the leading renormalon contribution which does not become smaller than  $\mathcal{O}(\Lambda_{\text{QCD}})$  [10]. This renormalon contribution is cancelled [3, 4] against the renormalon contribution [11] contained in the latter term. Therefore, after cancellation of the leading renormalons, the *genuine*  $\mathcal{O}(\alpha_S^3 m)$  part of the mass relation determines the accuracy of the present perturbation series relating the quark  $\overline{\text{MS}}$ -mass and the quarkonium spectrum.

As stated, for the binding energy the calculation including the *genuine*  $\mathcal{O}(\alpha_S^4 m)$  corrections has already been completed. Also, the “large- $\beta_0$  approximation” [12] is known to be a pragmatically feasible and empirically successful estimation method of the leading renormalon contributions. Taking these into account, one finds that it is sufficient<sup>†</sup> to calculate further the following two corrections in order to improve the accuracy of the spectrum by one order and to achieve a genuine accuracy at  $\mathcal{O}(\alpha_S^4 m)$ : (I) the  $\mathcal{O}(\alpha_S^4 m)$  relation between the  $\overline{\text{MS}}$ -mass and the pole-mass, and (II) the binding energy at  $\mathcal{O}(\alpha_S^5 m)$  in the large- $\beta_0$  approximation. This is because the leading renormalon contribution in the full  $\mathcal{O}(\alpha_S^5 m)$  correction to the binding energy will be incorporated by the large- $\beta_0$  approximation and the remaining part is expected to be irrelevant at  $\mathcal{O}(\alpha_S^4 m)$ .

Of these two corrections we calculate (II) analytically for the  $1S$ -state in this paper. Then we study its phenomenological applications. We can check validity of the above general argument explicitly at  $\mathcal{O}(\alpha_S^3 m)$  where we know the exact result. This will also be demonstrated.

Written in terms of the quark pole-mass  $m_{\text{pole}}$ , the mass of the quarkonium  $1S$ -state is given as a series expansion in the  $\overline{\text{MS}}$  coupling constant  $\alpha_S(\mu)$  defined in the theory with  $n_f$  massless quarks:

$$M_{1S} = 2m_{\text{pole}} - \frac{4}{9}\alpha_S(\mu)^2 m_{\text{pole}} \sum_{n=0}^{\infty} \left( \frac{\alpha_S(\mu)}{\pi} \right)^n P_n(L), \quad (1)$$

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\* See e.g. [8, 9] for introductory reviews of the subject.

<sup>†</sup> There exist other methods in which the renormalon contribution contained in the pole-mass is subtracted in certain approximations (e.g. [4]). In our opinion our method is a most natural one, embedded in the algorithm for higher order calculations of the whole spectrum; cancellation of infrared sensitivities in the whole spectrum follows from the fact that the quarkonium is a color-singlet small-size system.

where  $P_n(L)$  is an  $n$ -th-degree polynomial of  $L \equiv \log[3\mu/(4\alpha_S(\mu)m_{\text{pole}})]$ . At each order of the perturbative expansion  $c_n = P_n(0)$  represents a non-trivial correction, while the coefficients of  $L$ 's are determined by the renormalization-group equation.<sup>‡</sup> The polynomials relevant to our analysis read

$$P_0 = 1, \quad (2)$$

$$P_1 = \beta_0 L + c_1, \quad (3)$$

$$P_2 = \frac{3}{4}\beta_0^2 L^2 + \left(-\frac{1}{2}\beta_0^2 + \frac{1}{4}\beta_1 + \frac{3}{2}\beta_0 c_1\right) L + c_2, \quad (4)$$

$$P_3 = \frac{1}{2}\beta_0^3 L^3 + \left(-\frac{7}{8}\beta_0^3 + \frac{7}{16}\beta_0\beta_1 + \frac{3}{2}\beta_0^2 c_1\right) L^2 + \left(\frac{1}{4}\beta_0^3 - \frac{1}{4}\beta_0\beta_1 + \frac{1}{16}\beta_2 - \frac{3}{4}\beta_0^2 c_1 + \frac{3}{8}\beta_1 c_1 + 2\beta_0 c_2\right) L + c_3. \quad (5)$$

$\beta_n$ 's denote the coefficients of the QCD beta function given by

$$\beta_0 = 11 - \frac{2}{3}n_l, \quad \beta_1 = 102 - \frac{38}{3}n_l, \quad \beta_2 = \frac{2857}{2} - \frac{5033}{18}n_l + \frac{325}{54}n_l^2. \quad (6)$$

Note that, in the renormalization-group evolution, running of the coupling included in  $\log \alpha_S$  should be taken into account properly. This feature is unique to the perturbation series of a nonrelativistic boundstate spectrum. Presently  $c_n$ 's are known up to  $n = 2$  [1, 2]:

$$c_1 = \frac{97}{6} - \frac{11}{9}n_l, \quad (7)$$

$$c_2 = \frac{1793}{12} + \frac{2917\pi^2}{216} - \frac{9\pi^4}{32} + \frac{275\zeta_3}{4} + \left(-\frac{1693}{72} - \frac{11\pi^2}{18} - \frac{19\zeta_3}{2}\right)n_l + \left(\frac{77}{108} + \frac{\pi^2}{54} + \frac{2\zeta_3}{9}\right)n_l^2. \quad (8)$$

The aim of this paper is to calculate  $c_3$  in the large- $\beta_0$  approximation.

One might think that alternatively  $c_3$  may be estimated using the asymptotic form of the series expansion of the QCD potential at large orders. We could not, however, find a justification that  $c_3$  can be estimated with  $\mathcal{O}(\alpha_S^4 m)$  accuracy in this manner.

## 2 Outline of the calculation

In the large- $\beta_0$  approximation the wave functions and energy spectra (measured from  $2m_{\text{pole}}$ ) of quarkonium states are determined by solving the nonrelativistic Schrödinger equation

$$\left[\frac{\vec{p}^2}{m} + V_{\beta_0}(r)\right] \psi(\vec{x}) = E\psi(\vec{x}). \quad (9)$$

Here,  $m$  denotes the mass of the quark and it is irrelevant whether we use the pole-mass or the  $\overline{\text{MS}}$ -mass for  $m$  within our approximation; in particular it does not affect the leading

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<sup>‡</sup> For  $n \geq 3$ ,  $c_n$ 's include powers of  $\log \alpha_S$  unrelated to the renormalization group, i.e. which is not accompanied by  $\log \mu$  [13].

$n$	$R_n(\mu r)$
0	1
1	$2\ell$
2	$4\ell^2 + \frac{\pi^2}{3}$
3	$8\ell^3 + 2\pi^2\ell + 16\zeta_3$
$\vdots$	$\vdots$

Table 1: First few coefficients of the perturbative expansion of  $R(r)$ . Here,  $\ell = \log(\mu r) + \frac{5}{6} + \gamma_E$ .

renormalon cancellation.  $V_{\beta_0}$  denotes the QCD potential in the large- $\beta_0$  approximation given as a perturbation series [11]

$$V_{\beta_0}(r) = -C_F \frac{\alpha_S(\mu)}{r} \times R(r), \quad R(r) = \sum_{n=0}^{\infty} \left( \frac{\beta_0 \alpha_S(\mu)}{4\pi} \right)^n R_n(\mu r). \quad (10)$$

$C_F = 4/3$  is the color factor. First few coefficients  $R_n(\mu r)$  are listed in Table 1. To obtain  $S$ -wave solutions, we introduce a dimensionless variable  $z = C_F \alpha_S m r$  and set

$$\psi = \frac{1}{z} \exp \left[ - \int dz W(z) \right], \quad E = (C_F \alpha_S)^2 m \varepsilon. \quad (11)$$

Then the equation becomes

$$W' - W^2 - R/z = \varepsilon. \quad (12)$$

We expand  $W$ ,  $R$  and  $\varepsilon$  in perturbation series as in (10) and

$$W = \sum_{n=0}^{\infty} \left( \frac{\beta_0 \alpha_S}{4\pi} \right)^n W_n, \quad \varepsilon = \sum_{n=0}^{\infty} \left( \frac{\beta_0 \alpha_S}{4\pi} \right)^n \varepsilon_n, \quad (13)$$

and substitute them to (12). We find

$$W'_n - \sum_{k=0}^n W_k W_{n-k} - R_n/z = \varepsilon_n. \quad (14)$$

For the  $1S$ -state the zeroth-order solution is given by

$$W_0 = \frac{1}{2} - \frac{1}{z}, \quad \varepsilon_0 = -\frac{1}{4}, \quad (15)$$

and we may solve (14) recursively for  $n \geq 1$ :

$$\varepsilon_n = -\frac{1}{2} \int_0^\infty dt t^2 e^{-t} \left( \sum_{k=1}^{n-1} W_k(t) W_{n-k}(t) + R_n(t)/t \right), \quad (16)$$

$$W_n(z) = -\frac{e^z}{z^2} \int_z^\infty dt t^2 e^{-t} \left( \sum_{k=1}^{n-1} W_k(t) W_{n-k}(t) + R_n(t)/t + \varepsilon_n \right). \quad (17)$$

Thus,  $\varepsilon_n$ 's can be obtained by evaluating multiple integrals. We may render these integrals to forms which resemble Feynman-parameter integrals that appear in calculations of multi-loop Feynman diagrams. In this way we could use various techniques developed for Feynman diagram calculations.<sup>§</sup> We obtained the first two terms of the perturbative expansion as

$$\varepsilon_1 = -(\tilde{L} + 1), \quad (18)$$

$$\varepsilon_2 = -(3\tilde{L}^2 + 4\tilde{L} + 1 + \frac{\pi^2}{6} + 2\zeta_3), \quad (19)$$

where  $\tilde{L} = L + \frac{5}{6} = \log[\mu/(C_F\alpha_S m)] + \frac{5}{6}$ . From these we confirmed the corresponding parts of  $c_1$  and  $c_2$  in eqs. (7,8). We also obtained a new result:

$$\varepsilon_3 = -\left[8\tilde{L}^3 + 10\tilde{L}^2 + \left(\frac{4\pi^2}{3} + 16\zeta_3\right)\tilde{L} - 2 + \pi^2 + 16\zeta_3 + \frac{\pi^4}{90} - 2\pi^2\zeta_3 + 24\zeta_5\right]. \quad (20)$$

It follows that

$$c_3(\text{large-}\beta_0) = \beta_0^3 \left( \frac{517}{864} + \frac{19\pi^2}{144} + \frac{11\zeta_3}{6} + \frac{\pi^4}{1440} - \frac{\pi^2\zeta_3}{8} + \frac{3\zeta_5}{2} \right). \quad (21)$$

We note two points: (i) The result includes the level-5 zeta values ( $\zeta_5$  and  $\pi^2\zeta_3$ ). (ii) The result does not include  $\gamma_E$ , that is, all  $\gamma_E$  which appeared at intermediate stages of the calculation got cancelled. A more detailed description of our calculation will be published elsewhere.

### 3 Phenomenological applications

We examine the series expansion of the quarkonium  $1S$ -state spectrum numerically for the bottomonium and (remnant of) toponium. The input value for the coupling defined in the theory with 5 massless flavors ( $n_l = 5$ ) is  $\alpha_S^{(5)}(M_Z) = 0.119$ . We evolve the coupling and match it to the coupling of the theory with  $n_l = 4$  following [15]. Also we take the input values of the  $\overline{\text{MS}}$ -mass,  $\overline{m} \equiv m_{\overline{\text{MS}}}(m_{\overline{\text{MS}}})$ , and the pole-mass as  $\overline{m}_b = 4.20 \text{ GeV}/m_{b,\text{pole}} = 4.97 \text{ GeV}$  and  $\overline{m}_t = 165 \text{ GeV}/m_{t,\text{pole}} = 174.79 \text{ GeV}$ . The natural size of a quarkonium system is the Bohr radius. We define a corresponding scale parameter  $\mu_B$  such that

$$\mu_B = C_F\alpha_S(\mu_B)m_{\text{pole}} \quad (22)$$

holds. We examine the series expansion (1) with two different choices of the scale  $\mu$  in Table 2. The last terms (with stars) are evaluated using the value of  $c_3$  in the large- $\beta_0$  approximation (21). One sees that for the bottomonium the series expansions do not converge at all. For the toponium the series expansions converge very slowly. The numbers in square brackets represent estimates of the last terms using the asymptotic form of the series expansion of the potential  $V_{\beta_0}(r)$  [11]:

$$-C_F \frac{\alpha_S(\mu)}{r} \times R_n(\mu r) \sim -\frac{2e^{5/6}C_F\alpha_S(\mu)\mu}{\pi} \times 2^n n! \quad \text{for } n \gg 1. \quad (23)$$

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<sup>§</sup> We find the techniques developed in [14] particularly useful.

Series expansion of $M_{1S}$ [GeV]		
	$\mu = \mu_B$ [ expansion parameter = $\alpha_S(\mu_B)$ ]	$\mu = \overline{m}$ [ expansion parameter = $\alpha_S(\overline{m})$ ]
Bottomonium ( $n_l = 4$ )	$2 \times (4.97 - 0.11 - 0.12 - 0.20 - 0.25^*)$ [− 0.26]	$2 \times (4.97 - 0.06 - 0.08 - 0.11 - 0.15^*)$ [− 0.16]
Toponium ( $n_l = 5$ )	$2 \times (174.79 - 0.77 - 0.35 - 0.25 - 0.13^*)$ [− 0.14]	$2 \times (174.79 - 0.46 - 0.39 - 0.28 - 0.19^*)$ [− 0.25]

Table 2: Numerical evaluation of the series expansion of  $M_{1S}$ , eq. (1). Relevant parameters for the bottomonium are:  $\overline{m}_b = 4.20$  GeV,  $m_{b,\text{pole}} = 4.97$  GeV,  $\mu_B = 2.05$  GeV,  $\alpha_S^{(4)}(\mu_B) = 0.309$ ,  $\alpha_S^{(4)}(\overline{m}_b) = 0.229$ ; those for the toponium are:  $\overline{m}_t = 165$  GeV,  $m_{t,\text{pole}} = 174.79$  GeV,  $\mu_B = 32.9$  GeV,  $\alpha_S^{(5)}(\mu_B) = 0.1411$ ,  $\alpha_S^{(5)}(\overline{m}_t) = 0.1092$ .

The series approaches its asymptotic form faster when we choose  $\mu = \mu_B$ . These features are consistent with dominance of the leading renormalon contributions.

Next we rewrite the series expansion of  $M_{1S}$  in terms of the  $\overline{\text{MS}}$ -mass instead of the pole-mass. The leading renormalon contributions cancel in this case. Presently the relation between the  $\overline{\text{MS}}$ -mass and the pole-mass is known up to three loops [16, 17]. The only scale in this relation is the quark mass. Thus, we have two choices of scales,  $\overline{m}$  and  $\mu_B$ , in writing the series expansion of  $M_{1S}$ ; we take  $\mu = \overline{m}$  below. We eliminate the pole-mass completely and expand in  $\alpha_S(\overline{m})$ . We should properly take into account the fact that the renormalon contributions cancel between the terms whose orders in  $\alpha_S$  differ by one [18]. To this end we proceed as follows. We rewrite (1) as

$$\begin{aligned}
M_{1S} &= 2m_{\text{pole}} \times \left[ 1 + \sum_{n=2}^{\infty} Q_n \alpha_S(\overline{m})^n \right] \\
&= 2\overline{m} \times \left[ 1 + \sum_{n=1}^{\infty} d_n \alpha_S(\overline{m})^n \right] \times \left[ 1 + \sum_{n=2}^{\infty} Q_n \alpha_S(\overline{m})^n \right],
\end{aligned} \tag{24}$$

where  $Q_n$ 's are polynomials of  $\log[\alpha_S(\overline{m})]$  and  $d_n$ 's (the  $n$ -loop coefficients of the mass relation) are just constants independent of  $\alpha_S(\overline{m})$ . We identified  $Q_n \alpha_S^n$  as order  $\alpha_S^{n-1}$  and then reduced the last line to a single series in  $\alpha_S$ . Numerically we find<sup>¶</sup>

$$M_{1S} = 2 \times (4.20 + 0.36 + 0.13 + 0.040 + 0.0051^\sharp) \text{ GeV} \quad (\text{Bottomonium}), \tag{25}$$

$$M_{1S} = 2 \times (165.00 + 7.21 + 1.24 + 0.22 + 0.052^\sharp) \text{ GeV} \quad (\text{Toponium}). \tag{26}$$

The last terms (with sharps) are evaluated using the values of  $c_3$  and  $d_4$  [12] in the large- $\beta_0$  approximation. Convergences of the series improve markedly in comparison to those in Table 2. (Previous analyses similar to the one presented above can be found in [8, 7] and references therein.)

As we argued in Section 1, parametric accuracy of the last terms in (25,26) is  $\mathcal{O}(\alpha_S^4 m)$  and we need to know further only the exact value of  $d_4$  to make a perturbative evaluation

<sup>¶</sup> The values of  $d_1 \sim d_3$  are taken from eq. (14) of [17].

accurate up to this order (the exact form of  $c_3$  is not necessary). In order to verify validity of this argument, we replace  $c_2$  by its value in the large- $\beta_0$  approximation. Then the  $\mathcal{O}(\alpha_S^3 m)$  terms of (25,26) change to 0.043 and 0.22, respectively. Thus, we do not lose accuracy at this order by the replacement. On the other hand, if we replace  $c_2$  and  $d_3$  by their values in the large- $\beta_0$  approximation, the same terms change to 0.056 and 0.31, respectively. Thus, we lose the accuracy at  $\mathcal{O}(\alpha_S^3 m)$ . These aspects are consistent with our general argument. Also, they suggest that the last terms of (25,26) would be reasonable estimates of the orders of magnitude of the exact  $\mathcal{O}(\alpha_S^4 m)$  terms.

Finally we examine if we can use the asymptotic form of  $V_{\beta_0}(r)$ , eq. (23), to estimate  $c_3(\text{large-}\beta_0)$ . From the asymptotic values of the last terms for  $\mu = \mu_B$  in Table 2, we may extract approximate values of  $c_3(\text{large-}\beta_0)$  as

$$c_3(\text{asympt}) = \begin{cases} 2.55 \times 10^3 & (n_l = 4) \\ 1.98 \times 10^3 & (n_l = 5) \end{cases}, \quad (27)$$

while the corresponding values of  $c_3(\text{large-}\beta_0)$  are, respectively,  $2.46 \times 10^3$  and  $1.91 \times 10^3$ . If we substitute  $c_3(\text{asympt})$ , the last terms of (25,26) change to 0.0034 and 0.051, respectively. Therefore, we see that in the case of bottomonium the use of the asymptotic form does not reproduce our result in the large- $\beta_0$  approximation (25) with good (relative) accuracy. Presumably it is a sign of the next-to-leading renormalon contribution, which does not become smaller than  $\sim \Lambda_{\text{QCD}} \cdot (\Lambda_{\text{QCD}}/\mu_B)^2 \sim 2 \text{ MeV}$ , and which is not included in  $c_3(\text{asympt})$ . Namely, we conjecture that already the last term in eq. (25) stands close to the limit where an improvement of convergence of the perturbation series is possible by cancellation of the leading renormalon contributions.

## 4 Conclusions and discussions

We have calculated the  $\mathcal{O}(\alpha_S^5 m)$  correction to the quarkonium  $1S$ -state energy spectrum analytically in the large- $\beta_0$  approximation. As a result, in order to predict the  $1S$ -state spectrum at  $\mathcal{O}(\alpha_S^4 m)$  perturbative accuracy, only the four-loop relation between the  $\overline{\text{MS}}$ -mass and the pole-mass remains to be computed. Within the present approximation, the perturbation series of the bottomonium and toponium spectra show healthy convergent behaviors up to a genuine  $\mathcal{O}(\alpha_S^4 m)$  accuracy.

In the case of bottomonium, current theoretical uncertainty due to non-perturbative effects is estimated to be of the order of 0.1 GeV [6, 8]. It is much larger than the size of the last term of (25). We hope that in the future the theoretical uncertainty due to non-perturbative effects will be reduced by applications of e.g. lattice calculations or combinations of operator-product-expansion and sum rules.

Part of the *genuine*  $\mathcal{O}(\alpha_S^5 m)$  corrections have already been calculated [13]. Taking  $\mu = \overline{m}$ , their individual sizes are evaluated to be  $\Delta M_{1S} \sim \pm 0.05 \text{ GeV}$  both for the bottomonium and toponium; if we take  $\mu = \mu_B$ , they become even an order of magnitude larger. We do not know as yet whether it may indicate a breakdown of perturbative expansion of the spectrum at  $\mathcal{O}(\alpha_S^5 m)$  or a requisiteness for some new cancellation mechanism. It is also possible that

the sum of all the genuine  $\mathcal{O}(\alpha_s^5 m)$  corrections turns out to be much smaller. In any case, it would be better to separate the discussion of this problem from the determination of the genuine  $\mathcal{O}(\alpha_s^4 m)$  corrections (as we advocated in this paper), rather than to regard them as inseparable constituents of  $c_3$ .

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## References

- [1] A. Pineda and F. Ynduráin, Phys. Rev. **D58**, 094022 (1998).
- [2] K. Melnikov and A. Yelkhovsky, Phys. Rev. **D59**, 114009 (1999).
- [3] A. Hoang, M. Smith, T. Stelzer and S. Willenbrock, Phys. Rev. **D59**, 114014 (1999).
- [4] M. Beneke, Phys. Lett. **B434**, 115 (1998).
- [5] A. Penin and A. Pivovarov, Phys. Lett. **B435**, 413 (1998); Nucl. Phys. **B549**, 217 (1999).
- [6] A. Hoang, Phys. Rev. **D61**, 034005 (2000); M. Beneke and A. Signer, Phys. Lett. **B471** 233, (1999).
- [7] A. Hoang, et al., “*Top-Antitop Pair Production Close to Threshold: Synopsis of Recent NNLO Results*”, hep-ph/0001286.
- [8] M. Beneke, “*Perturbative heavy quark-antiquark systems*”, hep-ph/9911490.
- [9] Y. Sumino, “*Renormalon Cancellation in Heavy Quarkonia and Determination of  $m_b$ ,  $m_t$* ”, hep-ph/0004087.
- [10] M. Beneke and V. Braun, Nucl. Phys. **B426**, 301 (1994); I. Bigi, M. Shifman, N. Uraltsev and A. Vainshtein, Phys. Rev. **D50**, 2234 (1994).
- [11] U. Aglietti and Z. Ligeti, Phys. Lett. **B364**, 75 (1995).
- [12] M. Beneke and V. Braun, Phys. Lett. **B348**, 513 (1995).
- [13] N. Brambilla, A. Pineda, J. Soto and A. Vairo, Phys. Lett. **B470**, 215 (1999); B. Kniehl and A. Penin, Nucl. Phys. **B563**, 200 (1999); Nucl. Phys. **B577**, 197 (2000).
- [14] D. Broadhurst, hep-th/9604128; Eur. Phys. J. **C8**, 311 (1999).
- [15] K. Chetyrkin, B. Kniehl and M. Steinhauser, Phys. Rev. Lett. **79**, 2184 (1997).



- [16] K. Chetyrkin and M. Steinhauser, Phys. Rev. Lett. **83**, 4001 (1999); Nucl. Phys. **B573**, 617 (2000).
- [17] K. Melnikov and T. v. Ritbergen, Phys. Lett. **B482**, 99 (2000).
- [18] A. Hoang, Z. Ligeti and A. Manohar, Phys. Rev. Lett. **82**, 277 (1999); Phys. Rev. **D59**, 074017 (1999).